Хоробрых Анастасия

Parameterized Algorithms for Finding a Collective Set of Items

18.10.2023

Main definitions

- $N = [n]$ - set of agents
- $R = r_1, r_2, ..., r_m$ - set of items (resources)
- $u : R \to N$ - utility function
- $U = u_1, u_2, ..., u_n$ - preference profile
- bijection $u : R \to [m-1] \cup 0$ - Borda preference function.
  $i : r_1 \succ r_2 \succ \ \succ r_m$ - preference orders
- bijection $u : R \to \{0, 1\}^m$ - approval preference function(weighted approval preference functions, when $0, q$)

MKH

Definitions. Ordered Weighted Average Operators.

- For a k-item set $S = s_1, s_2, ..., s_k$ and preference function u, $\check{u}(S)$ - vector of utilities that u assigns to the items from S, sorted in nonincreasing order. $\dot{u}(S)$ - vector of items from S ordered nonincreasingly with respect to their utilities (ties are broken lexicographically).
- $\Lambda_k = (\lambda_1, \lambda_2, ..., \lambda_k)$ - vector of rational numbers
- $\sum_i \Lambda_k \cdot \check{u}_i(S)^T$ - OWA-score

MKH

Problem

Finding a k-item set with the highest possible OWA-score

OWA-WINNER problem. We are given a set $N = [n]$ of agents, a set R of m items, a preference profile U (consisting of n preference profiles), a positive integer $k < m$, an OWA-vector Λ_k, and an integer T. We ask whether there exists a k-item set $S \subseteq R$ such that $\Lambda_k - score_U(S) \geq T$.

MKH

Results

OWA-vector	utilities	complexity	ref.
$(1, 0, \ldots, 0)$	any	$\mathrm{FPT}(n)$	BSU
$(\underbrace{1, \ldots, 1}_{\pi}, 0, \ldots, 0)$	any	$\mathrm{FPT}(n, \pi)$ $\mathrm{XP}(n)$	Prop. 2 Prop. 8
$(\lambda_1, \overset{\pi}{\ldots}, \lambda_\pi, 0, \ldots, 0)$, for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\pi$	any	$\mathrm{FPT}(n, \pi)$	Prop. 2
$(\lambda_1, \ldots, \lambda_k)$, for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$	any	$\mathrm{FPT}(n, k)$	Thm. 1
$(0, \ldots, 0, 1)$	approval	$\mathrm{FPT}(n)$	Prop. 4
$(0, 1, 0 \ldots, 0)$	approval	$\mathrm{FPT}(n)$	Thm. 5
OWA-vector has ρ different values	approval	$\mathrm{FPT}(n, \rho)$	Thm. 5
OWA is piecewise non-increasing with ρ pieces	approval	$\mathrm{FPT}(n, \rho)$	Thm. 5
$(1, 2, 3, \ldots, k)$	approval	$\mathrm{FPT}(n)$	Thm. 6
$(\lambda_1, \ldots, \lambda_k)$ $\lambda_i \in \{-1, 0, 1\}, \forall i \in [k]$	approval	$\mathrm{W}[1]\text{-h.}(n)$	Thm. 7
$(0, 1, 0 \ldots, 0)$	Borda	$\mathrm{W}[1]\text{-h.}(n)$ $\mathrm{FPT\text{-}AS}(n)$	Thm 3, Thm. 10
$(1, 1/2, 1/3, \ldots, 1/k)$ $(\lambda_1, \ldots, \lambda_\pi, 0, \ldots, 0)$, for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\pi$	Borda	FPT-AS	Thm. 10

Table 1: Examples of the parametrized complexity of OWA-WINNER (it is NP-hard for all the listed settings; for vectors $(1, 0, \ldots, 0)$ this is due to Procaccia et al. (2008) and Lu and Boutilier (2011), for vectors $(1, 2, 3, \ldots)$ this is due to Faliszewski et al. (2018b), and for other vectors this is due

MKH

Arbitrary Utilities

Theorem 1. There is an FPT algorithm for OWA-WINNER with nonincreasing OWA vectors, parameterized by the number of agents and the size of the item set.

Theorem 1. Proof

1. Let $S = s_1, ..., s_k$ be a target k-item set.
2. Let for each agent i and each integer $j \in [k]$: $\psi_i(j) = l$ if $\dot{u}_i(S)[l] = s_j$. Permutations, which describe orders $\dot{u}_i(S)$
3. Let $h : [k] \to R$ - the matching between S and items from R.
4. Then $\Lambda_k - score(S) = \sum_i \sum_j \lambda_{\psi_i(j)} u_i(h(j))$

MKH

Theorem 1. Proof

Algorithm

1. For all possible guesses of $\psi_1, ..., \psi_n$ we will find maximizing h. ($k! \cdot n$ variants)
2. Let $w(r, j) = \sum_i \lambda_{\psi_i(j)} u_i(h(j))$.
3. Form a complete bipartite graph with weights $w(r, j)$
4. Find a maximum-weight matching in this graph. Time: $O(k \cdot |R|)$
5. Return “yes” if the weight of h is at least T .

Time: $O(k! \cdot n \cdot k \cdot |R|) = O(|R| \cdot f(n, k))$

MKH

Theorem 1. Proof

Correctness

1. Let $W = h(1), ..., h(k)$ be the set corresponding to h and let $\psi_1, ..., \psi_n$ be the permutations that lead to it
2. If $\forall i$ and $\forall j \neq j' \in [k]$ it holds that $\psi_i(j) < \psi_i(j') \to h(j)$ is ahead of $h(j')$ in $\dot{u}_i(W)$, then the weight of matching h in our bipartite graph equals the score of W.
3. Else $\exists i$ and $\exists j \neq j' \in [k]$ that $\psi_i(j) < \psi i(j')$ but $h(j')$ is ahead of $h(j)$ in $\dot{u}_i(W)$
4. Swap the values of $\psi_i(j)$ and $\psi_i(j')$ (and update the weights of the edges in our bipartite graph).
5. Since $\lambda_1 \geq \cdots \geq \lambda_k$, the new weight of h does not decrease and maximum does not increase. Repeat this process until we remove all inversions.

MKH

Borda utilities

$\beta(i)$-WINNER

OWA-WINNER problem with Borda preferences and OWA-vector of the form $(0, ..., 0, 1, 0, ..., 0)$, where first $i - 1$ values are 0.

Theorem 3. $\beta(i)$-WINNER is $W[1]$-hard when parameterized by the number of agents.

MKH

Theorem 3. Proof

Main idea Reduce from the $W[1]$-complete problem MULTICOLORED CLIQUE (parameterized by the solution size).

Notation

1. Let G be our input graph and let h be the number of colors ($h \geq 7$).
2. $V^{(i)} = \{v_1^i, \ldots, v_q^i\}$ - the set of vertices of colour i
3. $E(i, j)$ - the set of edges between colour j, i
4. $E_j(v)$ - the set of edges between v and vertices of colour j. We assume $|E_j(v)| \leq q - 1$
5. $H = C_h^2$

MKH

Theorem 3. Proof

Form an instance of $\beta(2)$-WINNER

1. $R = V(G) \cup E(G) \cup D$, where D - set of dummy items($|D|$ - polynomial)
2. $L = 4hq + 4Hq^2 + 2(h-1)^2q^2$
3. $D(l)$ listing l new dummy items that do not appear among the top L items in the other preference orders
4. $others$ listing the remaining items in some arbitrary order
5. $\vec{Y}$ - inverse order of that for Y
6. $k = h + H$ - item set size

Intention is that if there is a multicolored clique, then each highest-scoring item set consists of h vertex items and H edge items that form a clique.

MKH

Theorem 3. Proof

Form an instance of $\beta(2)$-WINNER

Vertex selection group

1. $\forall 1 \leq i < j \leq h$ *vertex selection group* contains agents $v_{i,j}$ and $v'_{i,j}$ with preference orders:

$$v_{i,j} : V_{(i)} \succ V_{(j)} \succ D(L-2q) \succ others$$

$$v'_{i,j} : \vec{V_{(i)}} \succ \vec{V_{(j)}} \succ D(L-2q) \succ others$$

MKH

Theorem 3. Proof

Form an instance of $\beta(2)$-WINNER

Edge selection group

1. $\epsilon = E(1,2), E(1,3), \ldots, E(1,h), E(2,1), \ldots$
2. ϵ has H elements
3. $\epsilon'(l)$ the partial preference order of the form $\epsilon(l) \succ D(q^2 - |\epsilon(l)|)$. Use always fresh set of dummy items
4. $\forall 1 \leq l < l' \leq H$ *edge selection group* contains agents $e_{l,l'}$ and $e'_{l,l'}$ with preference orders:

$$e_{l,l'} : \epsilon'(l) \succ \epsilon'(l') \succ D(L - 2q^2) \succ others$$

$$e'_{l,l'} : \epsilon'\vec{(l)} \succ \epsilon'\vec{(l')} \succ D(L - 2q^2) \succ others$$

MKH

Theorem 3. Proof

Form an instance of $\beta(2)$-WINNER

Consistency group

1. $S_j(v) = E_j(v) \succ D(q-1-|E_j(v)|) \succ v$
2. $\forall 1 \le i < j \le h$ *consistency group* contains agents $c_{i,j}$ and $c'_{i,j}$ with preference orders:

$$c_{i,j} : S_j(v_1^{(i)}) \succ \cdots \succ S_j(v_q^{(i)}) \succ D(L-q^2) \succ others$$

$$c'_{i,j} : S_j(v_q^{(i)}) \succ \cdots \succ S_j(v_1^{(i)}) \succ D(L-q^2) \succ others$$

3. $m = |R|$
4. $T = 2H(m - \frac{3}{2q} - \frac{1}{2}) + 2C_H^2(m - \frac{3}{2q^2} - \frac{1}{2}) + 2(h-1)^2(m - \frac{q}{2} - \frac{q^2}{2})$.

Proposition there is an item set with score at least T if and only if G contains a multicolored clique of size h.

MKH

Theorem 3. Proof

1. **Proper item set** - if for each color $i \in [h]$ it contains a single vertex from $V^{(i)}$, and for each pair of colors $1 \leq i < j \leq h$, it contains a single edge from $E^{(i,j)}$.
2. Calculate the score of proper item set X
3. **Proposition.** If X corresponds to a multicolored clique, then it has score T; otherwise its score is lower. Non-proper item sets have even lower scores.

MKH

Theorem 3. Proof

The score of proper item set X

1. $\forall v_{i,j}, v'_{i,j}$ - the second highest-ranked member of X is a vertex $v \in V^{(j)}$. Vertex selection agents assign exactly $m - q - t + m - q - (q - t + 1) = 2m - 3q - 1$ points to X
2. Similarly each pair of edge selection agents assigns utility $2m - 3q^2 - 1$ to X
3. $H(2m - 3q - 1) + C_H^2(2m - 3q^2 - 1) = 2H(m - \frac{3}{2q} - \frac{1}{2}) + 2C_H^2(m - \frac{3}{2q^2} - \frac{1}{2})$.

MKH

Theorem 3. Proof

1. Let u and e be the respective unique members of $V^{(i)}$ and $E^{(i,j)}$ included in X.
2. If u is incident to e, then both agents $c_{i,j}$ and $c'_{i,j}$ rank e ahead of u. u the second highest-ranked member of X, then they assign $m - q + m - q^2 = 2m - q - q^2$ to X
3. If u is not incident to e, then u is the second highest-ranked member of X according to exactly one of $c_{i,j}$ and $c'_{i,j}$, and e is the second highest-ranked member of X according to the other one. Then these agents assign score strictly lower than $2m - q - q^2$.
4. If X contains vertices and edges that form a multicolored clique, then its score is T, else its score is below T

MKH

Approval Utilities

Theorem 4. There is an FPT algorithm, parameterized by the number of agents, for αweighted-WINNER with OWAs of the form $(0, ..., 0, 1)$.

MKH