New bounds on the half-duplex communication complexity

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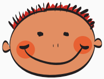
Communication models

Classical communication model

Introduced by Andrew Yao in 1979.

Alice

Bob



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$x \in \{0,1\}^n$

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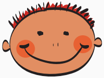

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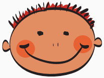

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Alice and Bob want to compute $f(x, y)$.

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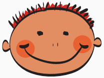
Alice $\longrightarrow$ Bob

b_1



$x \in \{0,1\}^n$ $y \in \{0,1\}^n$

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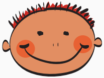
$\xrightarrow{\quad b_1 \quad}$

$\xrightarrow{\quad b_2 \quad}$

Bob



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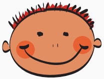
$\longrightarrow$ b_1

$\longrightarrow$ b_2

$\longleftarrow$ b_3

Bob



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$\xrightarrow{b_1}$

$\xrightarrow{b_2}$

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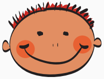

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Communication complexity of f is a minimal number of messages that is enough to compute f, denoted $D(f)$.

Half-duplex communication model

Players talk over half-duplex channel ("wakie-talkie") [HIMS18]

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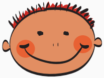
Bob



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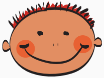

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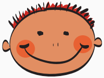

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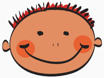

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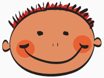
$\xleftarrow{\quad b_2 \quad}$



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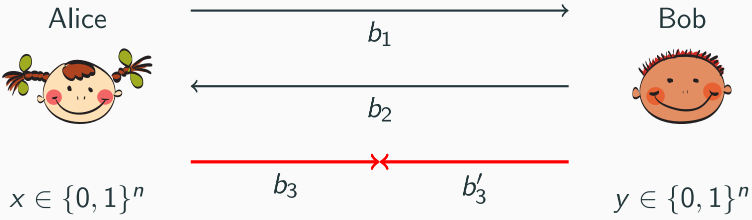

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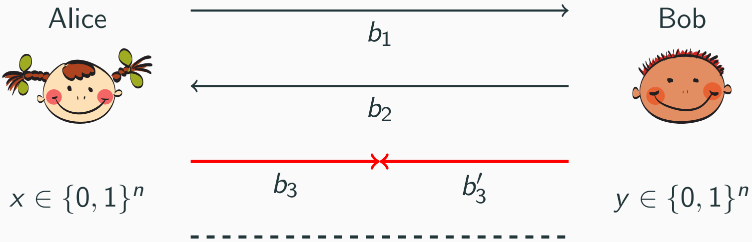

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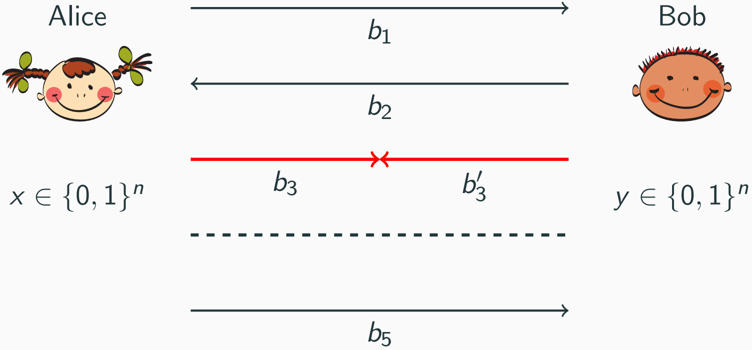

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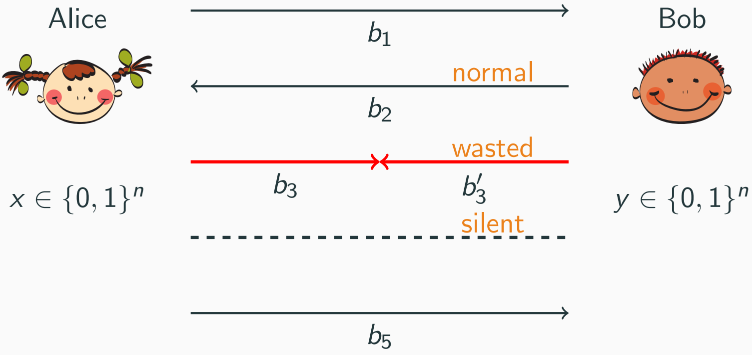

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Types of rounds

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1. Normal round: one player sends, other player receives.
2. Wasted round: both players send.
3. Silent round: both players receive.

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[HIMS18] considered three variants of how silent rounds work.

- **Half-duplex with silence**: the players receive some special symbol (i.e., silence), neither 0 nor 1.
- **Half-duplex with zero**: the players receive 0 (indistinguishable from normal round).
- **Half-duplex with an adversary**: the players receive bits chosen by an adversary (or some noise).

Basic bounds

For every $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ the following holds.

1. $D_s^{hd}(f) \leq D_0^{hd}(f) \leq D_a^{hd}(f) \leq D(f)$.

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3. $D(f)/3 \le D^{hd}_s(f)$
   (half-duplex communication with silence can be simulated by a classical protocol sending three bits per each round of the original protocol).

Note that multiplicative constants are important.

Our results

Communication problems

We study complexity of the following communication problems.

- Disjointness: $\mathrm{DISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, such that $\mathrm{DISJ}_n(x,y) = 1 \iff \forall i : x_i = 0 \vee y_i = 0$.
- Karchmer-Wigderson game for $\mathrm{MOD}p$ function defined by $\mathrm{MOD}p(x) = 0 \iff x_1 + \ldots + x_n = 0 \mod p$,
- Karchmer-Wigderson game for RecMaj function defined by

$$\mathrm{RecMaj}_n(x_1,\ldots,x_n) = \mathrm{Maj}_3\big(\mathrm{RecMaj}_{\frac{n}{3}}(x_1,\ldots,x_{\frac{n}{3}}),\\ \mathrm{RecMaj}_{\frac{n}{3}}(x_{\frac{n}{3}+1},\ldots,x_{\frac{2n}{3}}), \mathrm{RecMaj}_{\frac{n}{3}}(x_{\frac{2n}{3}+1},\ldots,x_n)\big),$$

The Karchmer–Wigderson game for $f\colon \{0,1\}^n \to \{0,1\}$:
Alice is given $x \in f^{-1}(0)$, Bob is given $y \in f^{-1}(1)$, and they want to find an $i \in [n]$ such that $x_i \neq y_i$.

Summary of results

	EQ_n	IP_n	DISJ_n	$\mathrm{KW}_{\mathrm{MOD2}}$	
D^{hd}_s	$\geq n/\log 5$ $\leq n/\log 5 + o(n)$	$\geq n/1.67$	$\geq n/\log 5$ $\leq n/2 + O(1)$	$\geq 1.12 \log n$ $\leq 1.262 \log n$	$\star$
D^{hd}_0	$\geq n/\log 3$ $\leq n/\log 3 + o(n)$	$\geq n/1.234$	$\geq n/\log 3$ $\leq 3n/4 + o(n)$	$\geq 1.62 \log n$ $\leq 1.893 \log n$	$\star$
D^{hd}_a	$\geq n/\log 2.5$	$\geq n$	$\geq n/\log 2.5$	$= 2 \log n$	$\star$

Other bounds:

$$\mathrm{D}^{hd}_s(\mathrm{KW}_{\mathrm{MOD3}}) \leq 1.893 \log n, \quad \mathrm{D}^{hd}_s(\mathrm{KW}_{\mathsf{RecMaj}}) \leq 2 \log_3 n,$$
$$\mathrm{D}^{hd}_s(\mathrm{KW}_{\mathrm{MOD5}}) \leq 2.46 \log n, \quad \mathrm{D}^{hd}_0(\mathrm{KW}_{\mathsf{RecMaj}}) \leq 2 \log_3 n,$$
$$\mathrm{D}^{hd}_s(\mathrm{KW}_{\mathrm{MOD11}}) \leq 3.48 \log n.$$

For arbitrary $p \geq 7$, $\mathrm{D}^{hd}_s(\mathrm{KW}_{\mathrm{MOD}p}) \leq 1.16 \left\lceil 1 + \log_3 \frac{p}{2} \right\rceil \cdot \log n$.

For arbitrary $p > 2$, lower bounds $(\star)$ applies to $\mathrm{KW}_{\mathrm{MOD}p}$.

Non-deterministic communication complexity

We introduce non-deterministic half-duplex communication complexity based on an alternative definition of classical non-deterministic complexity.

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We prove bound relating it to the classical non-deterministic complexity.

For any function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, we show that

$$\mathrm{N}_s^{hd}(f) = \mathrm{N}(f)/\log 5 + \Theta(\log \mathrm{N}(f)),$$
$$\mathrm{N}_0^{hd}(f) = \mathrm{N}(f)/\log 3 + \Theta(\log \mathrm{N}(f)),$$
$$\mathrm{N}_a^{hd}(f) \geq \mathrm{N}(f)/\log 3,$$

Highlights of the proofs

Upper bound on $\mathrm{D}_0^{hd}(\mathrm{DISJ})$

We start with proving a weaker bound.

Lemma

For all $n \in \mathbb{N}$, $\mathrm{D}_0^{hd}(\mathrm{DISJ}_n) \leq 5n/6 + O(\log n)$.

- The players split input strings into blocks of length 2.
- [Phase 1] The players spend $n/2$ rounds to compare all blocks.
- For some blocks the situation is not clear.
- [Phase 2] Send additional information for every pair of blocks that was processed in a silent round.

Note that we need some case analysis to ensure that there will be at most $n/3$ silent rounds.

Upper bound on $\mathrm{D}_0^{hd}(\mathrm{DISJ})$ (contd.)

Theorem

For all $n \in \mathbb{N}$, $\mathrm{D}_0^{hd}(\mathrm{DISJ}_n) \leq 3n/4 + o(n)$.

- The players split input strings into blocks of length 2.
- [Phase 1] The players spend $n/2$ rounds to compare all blocks.
- For some blocks the situation is not clear.
- [Phase 2] Compose new inputs from all the blocks processed in silent rounds and run the protocol recursively.

Given that the number of silent rounds is at most $n/3$ we get

$$\mathrm{D}_0^{hd}(\mathrm{DISJ}_n) \leq \sum_{i=0}^{\lceil \log_3(n) \rceil} \frac{n}{2 \cdot 3^i} + o(n) \leq \frac{3n}{4} + o(n).$$

Theorem
For any $p \geq 2$,

$$\mathrm{D}_s^{hd}(\mathrm{KW}_{\mathrm{MOD}p}) > 1.12 \log n,$$
$$\mathrm{D}_0^{hd}(\mathrm{KW}_{\mathrm{MOD}p}) > 1.62 \log n,$$
$$\mathrm{D}_a^{hd}(\mathrm{KW}_{\mathrm{MOD}p}) \geq 2 \log n - O(1).$$

- There is a probability distribution over the inputs:
  - at the beginning each player has uncertainty roughly $\log n$ bits about the input of the
  - at the end each player knows the input of the other player.
- Each player learns roughly $\log n$ bits of information.
- We upper bound the amount of information the players can learn in one round for all the half-duplex models.

Open problems

1. Is there any $\alpha < 1$ such that for any function f, $D_0^{hd}(f) \leq \alpha n + o(n)$?
2. Is there any function f, such that at the same time $D(f) \geq n - o(n)$ and $D_a^{hd}(f) \leq \alpha n + o(n)$ for some $\alpha < 1$.
3. Prove new lower bound for disjointness using information-theoretic methods.
4. Prove an upper bound on $\mathrm{D}_0^{hd}(\mathrm{KW}_{\mathrm{MOD}p})$.
5. Prove better upper bound on D_s^{hd}(RecMaj).
6. Upper bounds on IP_n in all the models.

Thanks for your attention!

Upper bounds on the Karchmer-Wigderson games

Ternary search

- $\mathrm{D}_s^{hd}(\mathrm{KW}_{\mathrm{MOD2}}) \leq 2\log_3 n + O(1) < 1.262 \log n$.
- $\mathrm{D}_0^{hd}(\mathrm{KW}_{\mathrm{MOD2}}) \leq 3\log_3 n + O(1) < 1.893 \log n$.
- $\mathrm{D}_s^{hd}(\mathrm{KW}_{\mathsf{RecMaj}}) \leq 2\log_3 n$.
- $\mathrm{D}_0^{hd}(\mathrm{KW}_{\mathsf{RecMaj}}) \leq 2\log_3 n$.

Binary search + encoding [Chin90]

- $\mathrm{D}_s^{hd}(\mathrm{KW}_{\mathrm{MOD5}}) \leq 2.46 \log n$.
- $\mathrm{D}_s^{hd}(\mathrm{KW}_{\mathrm{MOD11}}) \leq 3.48 \log n$.
- For all $p \geq 7$, $\mathrm{D}_s^{hd}(\mathrm{KW}_{\mathrm{MOD}p}) \leq 1.16 \left\lceil 1 + \log_3 \frac{p}{2} \right\rceil \cdot \log n$.